Dynamical Disentanglement across a Point Contact in a Non-Abelian Quantum Hall State

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We analyze tunneling of non-Abelian quasiparticles between the edges of a quantum Hall droplet at Landau level filling fraction $\nu = 5/2$, assuming that the electrons in the first excited Landau level organize themselves in the non-Abelian Moore-Read Pfaffian state. We formulate a bosonized theory of the modes at the two edges of a Hall bar; an effective spin-$1/2$ degree of freedom emerges in the description of a point contact. We show how the crossover from the high-temperature regime of weak quasiparticle tunneling between the edges of the droplet, with 4-terminal $R_{xx} \sim T^{-3/2}$, to the low-temperature limit, with $R_{xx} - \frac{1}{10}\frac{h}{e^2} \sim -T^4$, is closely related to the two-channel Kondo effect. We give a physical interpretation for the entropy of $\ln(2\sqrt{2})$ which is lost in the flow from the ultraviolet to the infrared.

Introduction. There is indirect evidence [1] that a non-Abelian topological state may occur at the observed quantized Hall plateau with $\sigma_{xy} = \frac{5}{2}\frac{e^2}{h}$ [2, 3]. The leading candidate is the Moore-Read Pfaffian state [4, 5], in which charge-$e/4$ quasiparticles exhibit non-Abelian braiding statistics [6, 7, 8, 9, 10, 11, 12]. This state would support topologically-protected qubits, whose observation would confirm the non-Abelian nature of this quantized Hall plateau [13]. Other interference measurements would also directly probe the non-Abelian braiding statistics of quasiparticles in this state [8, 14, 15, 16].

All of these proposed experiments suggest employing the gapless edge excitations as a probe of the bulk. Interedge tunneling of quasiparticles occurs when two edges of the Hall bar are brought into close proximity. In these proposals, it is assumed that tunneling at these contacts is weak. However, just as in the Abelian states [17, 18, 19], as the temperature and voltage are decreased, the effective quasiparticle tunneling strength increases until the Hall droplet is effectively split into two at the point contact.

In this paper, we describe this crossover precisely. The non-Abelian statistics of the quasiparticles means that care is required to even define the tunneling operator at the point contact. Although the tunneling occurs at a single point in space, the quasiparticle which is tunneling can be entangled with others far away. We find the tunneling operator for the simplest kind of point contact in the Moore-Read state in terms of the underlying edge theory, a critical 2d Ising model (the neutral sector) and a free boson (the charged sector). This allows us to describe the perturbation expansion of quasiparticle tunneling processes at the point contact in terms of the chiral correlators of the edge theory. We show how to bosonize the tunneling operator, and find the remarkable result that quasiparticle transport through the point contact is closely related to the two-channel anisotropic Kondo effect [22] and resonant tunneling in Luttinger liquids [18].

As the droplet breaks in two, the initial entropy, corresponding to the uncertainty in the non-Abelian topological charges of the two halves of the droplet, is removed. We show how the

difference of entropies between the system without tunneling (the ultraviolet limit) and the system split into two droplets (the infrared limit) is $S_{UV} - S_{IR} = \ln(2\sqrt{2})$. When only the most relevant tunneling operator is present, the arrival at the infrared strong-coupling fixed point is fine-tuned so that the leading irrelevant operator has scaling dimension 8 and tunnels a *pair* of electrons from one half of the droplet to the other. When a subleading marginal tunneling operators is present in the ultraviolet, the leading irrelevant operator in the infrared has scaling dimension 3 and tunnels a single electron.

Edge Excitations at $\nu = 5/2$. We assume that the lowest Landau level (of both spins) is filled and the first excited Landau level is in the Moore-Read state. There will, therefore, be two integer quantum Hall edge modes, which will be the outermost excitations of the system. Since we will be focussing on tunneling across the interior of a Hall droplet, these modes will participate very weakly. Hence, we will ignore them and focus on the half-filled first excited Landau level. The gapless chiral theory describing the edge excitations has both a charged sector and a neutral sector. The charged sector is described by a free boson ϕ_c. Edge modes described solely by free bosons are characteristic of abelian fractional quantum Hall states, and are now very well understood [17, 18, 19]. The novel properties of the non-Abelian state believed to be realized at $\sigma_{xy} = \frac{5}{2}\frac{e^2}{h}$ arise from the neutral sector, which is described by a critical Ising field theory [20].

The chiral part of the $1 + 1$ dimensional Ising field theory contains a spin field σ and a free Majorana fermion ψ. Correlators involving only the fermion are trivial to compute, but finding those involving the Ising spin field requires much more work. One reason the physics of the Moore-Read state is so interesting is because the operator which creates or annihilates a charge $e/4$ quasiparticle contains the chiral part of the spin field. The spin field is not local with respect to the Majorana fermion: it creates a branch cut for the fermion field. When a charge $e/4$ quasiparticle tunnels to or from an edge of a droplet, the fermion boundary conditions around the Hall droplet change: they are anti-periodic when there is an even number of $e/4$ quasiparticles in the bulk, and periodic when
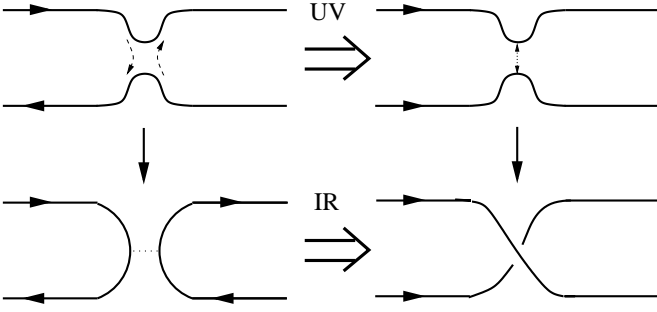
FIG. 1: We ignore the $\nu = 2$ edge excitations of the filled lowest Landau level (of both spins) and focus on the $\nu = 1/2$ edge excitations. To compute their correlation functions, we redraw the lower edge so that its excitations also propagate to the right. This allows us to bosonize the neutral fermionic excitations of both edges.

there is an odd number.

We study quasiparticle tunneling between two different points on the edge of a Hall droplet. The quasiparticle of charge $e/4$ is created by the operator $\sigma e^{i\phi_c/(2\sqrt{2})}$, where the chiral boson ϕ_c is normalized so that $e^{ia\phi_c}$ has dimension $a^2/2$. A chiral Ising spin field has dimension $1/16$, so the operator describing tunneling of one such quasiparticle from one edge to the other has dimension $1/4$. Other tunneling processes are described by the dimension-$1/2$ operator tunneling the charge-$e/2$ quasiparticle created by $e^{i\phi_c/\sqrt{2}}$, and the marginal operator tunneling the neutral fermionic quasiparticle ψ. The tunneling Lagrangian is schematically

$$\mathcal{L}^{\text{tun}} = \lambda_{1/4}\,\sigma_\alpha\sigma_\beta\, e^{i(\phi_{c\alpha}-\phi_{c\beta})/2\sqrt{2}} + \text{h.c.}$$
$$+ \lambda_{1/2}\, e^{i(\phi_{c\alpha}-\phi_{c\beta})/\sqrt{2}} + \text{h.c.} + i\lambda_1\psi_\alpha\psi_\beta \quad (1)$$

where the subscripts α, β refer to the spatial points on either side of the point contact, as in Fig. 1. We say "schematically" because as we will describe below, the operator $\sigma_\alpha\sigma_\beta$ is non-local and requires additional information to be defined precisely. In the limit that $\lambda_{1/4}$ is small, we can ignore such complications to compute the leading-order behavior [21]. We expect $\lambda_{1/2} \ll \lambda_{1/4}$ (because $\lambda_{1/2} \sim \lambda_{1/4}^2$), in which case the 4-terminal longitudinal resistance (defined as the voltage drop along one edge of the Hall bar divided by the transmitted current) scales as

$$R_{xx} \sim \lambda_{1/4}^2\, T^{-3/2}. \quad (2)$$

At finite voltage $V > T$, we instead have $I \sim V^{-1/2}$.

Chiral Correlation Functions. Since the tunneling amplitudes $\lambda_{1/4}$ and $\lambda_{1/2}$ increase in importance as the temperature decreases (2), the weak-tunneling regime does not extend to arbitrarily low temperatures. Therefore, we must go beyond the lowest-order perturbation theory contribution given in (2). This requires understanding correlators of chiral Ising fields; more information than just the space-time locations of the fields must be provided in order to define correlators uniquely. For example, the four-point function of non-chiral Ising spin

fields does not decompose into a simple product of a chiral correlator times its antichiral conjugate, but rather is a sum of such terms [24]. In other words, local Ising fields cannot simply be decomposed into a product of a left-moving part times a right-moving part. In the Moore-Read state, this is a manifestation of the non-Abelian statistics of the quasiparticles.

To understand the situation in more depth, note that a pair of quasiparticles can be in either of two topologically distinct states [4, 6, 7, 9], the two states of the qubit which they form [13]. A simple physical picture arises from considering the closely-related system of a superconductor with pairing symmetry $p + ip$. Charge-$e/4$ quasiparticles correspond to half-flux quantum vortices in this superconductor; there is a Majorana fermion zero mode in the core of each such vortex [9]. The Majorana fermion zero modes associated with two vortices can be combined into a Dirac fermion zero mode, which can be either occupied or unoccupied – the two states $|0\rangle$ and $|1\rangle$ of the qubit. In this way [9], one can show that there are 2^{n-1} states of $2n$ quasiparticles [6, 7]. When quasiparticles are braided, these states are rotated into each other according to the spinor representation of $SO(2n)$ [6, 10].

The same issue arises in a correlation function of $2n$ quasiparticle operators. There is a 2^{n-1}-dimensional vector space of such correlation functions – called *conformal blocks* [24] – which transform into each other as the quasiparticle positions are taken around each other in $1 + 1$-dimensional spacetime. This is a consequence of the fact that the operator product of two spin fields contains both the identity field and the fermion; this "fusion" is written schematically as $\sigma \cdot \sigma \sim I + \psi$. Combining the two languages, we see that when a pair of quasiparticles is in the state $|0\rangle$, they fuse to I; when they are in the state $|1\rangle$, they fuse to ψ.

To define precisely the tunneling operator and its conformal blocks, we must therefore also specify the state of the qubit associated with the quasiparticle which tunnels and the quasihole which it leaves behind. We assume that tunneling processes do not create additional neutral fermions in the bulk because the topological state of the system cannot be affected by a tiny motion (tunneling from one side of the point contact to the other) of a quasiparticle. This means that when a quasiparticle tunnels from one edge to another, it forms a qubit in state $|0\rangle$ with the quasihole which is left behind. This assumption fixes uniquely the conformal blocks occuring in the perturbative expansion of (1) in power of $\lambda_{1/4}$. The term with n tunneling operators is an element of a 2^{n-1} dimensional vector space of conformal blocks. We label the basis elements of this vector space by $[m_1, m_2, ..., m_n]$ with $m_i = 0$ or 1. This means that the ith pair $\sigma_\alpha\sigma_\beta$ fuses to I if $m_i = 0$ and to ψ if $m_i = 1$. Our assumption means that a formal expression such as $\langle \sigma_\alpha\sigma_\beta\, \sigma_\alpha\sigma_\beta \ldots \rangle$ arising in the perturbative expansion is given by the conformal block $[0, 0, \ldots]$.

Cluster decomposition. Provided the measurement time scale is short compared to the time for an excitation to propagate around the droplet from one side of the contact to the other, one can treat the two sides of the point contact as two independent edges (so σ_α and σ_β as well as $\phi_{c\alpha}$ and $\phi_{c\beta}$ are

independent fields). One can then use cluster decomposition to break each correlator into a product of correlators in two separate models, e.g.

$$\langle \sigma_\alpha \sigma_\beta \, \sigma_\alpha \sigma_\beta \ldots \rangle \sim \langle \sigma_\alpha \, \sigma_\alpha \ldots \rangle_\alpha \; \langle \sigma_\beta \, \sigma_\beta \ldots \rangle_\beta \qquad (3)$$

The complication is that the left-hand-side of (3) is specified by how each $\sigma_\alpha(\tau_a)$ fuses with $\sigma_\beta(\tau_a)$ (with τ_a the imaginary time of the ath tunneling event), but we would like to treat σ_α and σ_β on the right-hand-side as independent fields. We thus need to disentangle the fields σ_α and σ_β in order to break the left-hand-side into the product of two independent correlators, and then specify how neighboring σ_α's fuse with each other in $\langle \sigma_\alpha \, \sigma_\alpha \ldots \rangle_\alpha$ and neighboring σ_β's fuse in $\langle \sigma_\beta \, \sigma_\beta \ldots \rangle_\beta$.

To overcome these complications, we utilize the work of Moore and Seiberg, where relations among conformal blocks are derived by utilizing a variety of consistency conditions [25]. Because of the non-abelian structure, braiding (exchanging the order of operators) not only results in phases, but can change the fusion channels as well. In this way we can change from the basis in which each $\sigma_\alpha(\tau_a)$ fuses in a definite way with its neighboring $\sigma_\beta(\tau_a)$, to a basis in which each $\sigma_\alpha(\tau_{2j-1})$ fuse in a definite way with $\sigma_\alpha(\tau_{2j})$, and similarly for the σ_β's. Letting $\mu = \alpha$ or β, we label this new basis by $(m_1, m_2, \ldots, m_{n/2})_\mu$ where $m_j = 0, 1$ when $\sigma_\mu(\tau_{2j-1})$ and $\sigma_\mu(\tau_{2j})$ fuse to be into the I and ψ channels. We find [23]

$$[0, 0, \ldots, 0] = \sum_{\{m_j\}=0,1} \prod_{\mu=\alpha,\beta} (m_1, m_2, \ldots, m_{n/2})_\mu . \qquad (4)$$

With (1) and (4), we now have unambiguous expressions for the dynamics of the point contact, treating the two sides as independent edges. The conformal blocks $(m_1, m_2, \ldots)_\mu$ can be computed by solving differential equations [24]. However, the results are quite complicated, and it is difficult to extract much intuition from them. Therefore, it is useful to express them in a different fashion.

Bosonized Formulation. Typically, one treats ψ_α and ψ_β as the right- and left-handed parts of a single non-chiral theory. However, it is more convenient here to treat them as having the same chirality by flipping the chirality of one edge, as in Fig. 1. Then, the two chiral Majorana fermion fields form a single chiral Dirac fermion field, which can be bosonized, $\psi_\alpha + i\psi_\beta \sim e^{i\phi_\sigma}$. Here the chiral bosonic field ϕ_σ is normalized so that $e^{ia\phi_\sigma}$ has dimension $a^2/2$. With this bosonization scheme a semiclassical (instanton) analysis is possible for the the point contact dynamics in the IR limit. Moreover, it enables us to exploit similarities with the problem of resonant tunneling between Luttinger liquids [18], the Kondo problem [22], and dissipative quantum mechanics [26].

To bosonize our chiral correlators of spin fields we use the methods of Ref. 25. The result is: [23]

$$\prod_{\mu=\alpha,\beta} (m_1, m_2, \ldots, m_{n/2})_\mu = \left\langle \prod_{j=1}^{n/2} \left(e^{i\left(\phi_\sigma(\tau_{2j-1}) - \phi_\sigma(\tau_{2j})\right)/2} \right. \right.$$
$$\left. \left. + (-1)^{m_j} e^{-i\left(\phi_\sigma(\tau_{2j-1}) - \phi_\sigma(\tau_{2j})\right)/2} \right) \right\rangle \qquad (5)$$

Performing the sums in (4) gives the nice result:

$$[0, 0, \ldots, 0] = \left\langle \prod_{j=1}^{n/2} e^{i\left(\phi_\sigma(\tau_{2j-1}) - \phi_\sigma(\tau_{2j})\right)/2} \right\rangle . \qquad (6)$$

Remarkably, the right-hand-side is the same as $\left\langle \left(S^+ e^{-i\phi_\sigma/2} + S^- e^{i\phi_\sigma/2} \right) \left(S^+ e^{-i\phi_\sigma/2} + S^- e^{i\phi_\sigma/2} \right) \ldots \right\rangle$ where $\vec{S}$ is a single spin-$1/2$ degree of freedom. Consequently, the perturbation expansion of $\mathcal{L}^{\text{tun}}$ in eqn. (1) is identical to that of

$$\tilde{\mathcal{L}}^{\text{tun}} = \lambda_{1/4} \left(S^+ e^{-i\phi_\sigma/2} + S^- e^{i\phi_\sigma/2} \right) \cos(\phi_\rho/2)$$
$$+ \lambda_{1/2} \cos\phi_\rho + \lambda_1 \, \partial_x \phi_\sigma, \qquad (7)$$

with a charge boson $\phi_\rho \equiv \left(\phi_{c\alpha} - \phi_{c\beta} \right)/\sqrt{2}$. Thus, in translating chiral correlation functions of quasiparticle tunneling operators into bosonic language, we see the emergence of an effective spin-$1/2$ degree of freedom. This shows that the dynamics of a point contact in a Moore-Read non-Abelian quantum Hall state is a variant of the two-channel Kondo problem and equivalent to resonant tunneling between two $g = 2$ Luttinger liquids. Upon setting $\phi_\rho = 0$ (appropriate for a $p+ip$ superconductor), the neutral sector is literally the single-channel (anisotropic) Kondo Hamiltonian.

Kondo Crossover. Consider, first, the case in which only the most relevant tunneling amplitude, $\lambda_{1/4}$, is non-zero. In this case the Hamiltonian in (7) is invariant under $\phi_\sigma \to -\phi_\sigma$ together with a π rotation of the spin about the $x-$axis, a Kramers-Wannier duality symmetry for the non-chiral Ising model. Under the replacement $\phi_{\rho,\sigma} \to 2\phi_{\rho,\sigma}$, the tunneling term is identical to the two-channel Kondo model with $J_z = 0$, $J_{x,y} = \lambda_{1/4}$. By performing a unitary transformation, $U = \exp(iS^z \phi_\sigma/2)$, the tunneling Hamiltonian becomes:

$$U \, \mathcal{H}_{\text{tun}} \, U^\dagger = \lambda_{1/4} S^x \cos(\phi_\rho/2) + \pi v_n S^z \partial_x \phi_\sigma. \qquad (8)$$

The Toulouse [22] limit corresponds to dropping the final term. Although the presence of this marginal perturbation does change the dimensions of the operators in the UV, it does not greatly effect physical properties in the IR. To see this, note that $\lambda_{1/4}$ is strongly relevant and grows in the IR, so that the energy is minimized with $S^x = +1/2$, $\phi_\rho = 2(2n+1)\pi$ or $S^x = -1/2$, $\phi_\rho = 2(2n)\pi$. In this limit, the charge mode is completely reflected, so that $R_{xx} = \frac{1}{10} \frac{h}{e^2}$. (This peculiar value is the voltage drop due to the complete backscattering of the $\nu = 1/2$ edge divided by the transmitted current carried by the $\nu = 2$ edges.) At any of these minima, $\langle S^z \rangle = 0$, and fluctuations in S^z can be integrated out, generating terms such as $(\partial_x \phi_\sigma)^2 \sim \psi_1 \partial_x \psi_1 + \psi_2 \partial_x \psi_2$ (which does not couple the two edges) and $(\partial_x \phi_\sigma)^4 \sim \psi_1 \partial_x \psi_1 \, \psi_2 \partial_x \psi_2$, which couples the energies of the fermionic modes at the two edges.

Irrelevant perturbations at the infrared fixed point correspond to instantons connecting the minima of (8). The instanton $\Delta\phi_\rho = \pm 4\pi$, $\Delta\phi_\sigma = 0$, $\Delta\mathbf{S} = 0$ corresponds to the operator,

$$H_{\text{pair}}^{\text{tun}} = v_2 \cos(4\phi_\rho), \qquad (9)$$

which tunnels a charge-2 boson between the two droplets. When $\lambda_1 = 0$ in the UV, duality implies that this dimension-8 operator is the leading irrelevant tunneling operator in the IR, giving $R_{xx} - \frac{1}{10}\frac{h}{e^2} \sim -v_2^2\, T^{14}$ as $T \to 0$.

With $\lambda_1 \neq 0$, the tunneling Hamiltonian in the UV is no longer invariant under Kramers duality, $\phi_\sigma \to -\phi_\sigma$. The operator that tunnels electrons between the two drops in the IR,

$$H_{\mathrm{el}}^{\mathrm{tun}} = v_1\, \partial_x\phi_\sigma\, \cos(2\phi_\rho) = v_1\, i\psi_1\psi_2\, \cos(2\phi_\rho), \quad (10)$$

is then not forbidden by symmetry. In this generic case, the leading low-temperature correction to complete backscattering at the contact is determined by this dimension-3 electron tunneling operator: $R_{xx} - \frac{1}{10}\frac{h}{e^2} \sim -v_1^2\, T^4$.

Entropy Loss. The two states of the spin one-half degree of freedom that emerges in the weak-tunneling limit, $S^z = \pm 1/2$, correspond physically to whether there is an even or an odd number of bulk charge-$e/4$ quasiparticles to the left (say) of the point contact. In the infrared limit this spin is screened. Therefore, ground state entropy is lost in the flow. In the absence of the charged mode (as for a $p + ip$ superconductor), the model is equivalent to the single-channel Kondo problem. The entropy loss here is therefore simply $\ln 2$, since a spin-$1/2$ degree of freedom in the UV is completely screened in the IR. With the charged mode present, the entropy loss is larger. This follows from the Toulouse limit of Eq. (8). With $v_n = 0$, one can set $S^x = 1$, and the problem reduces to a boundary sine-Gordon model (a pure $\cos(a\phi)$ boundary perturbation). The entropy loss for a boundary sine-Gordon tunneling operator of dimension $a^2/2$ is $-\ln(a^2/2)/2$ [27]. Therefore the entropy loss in our full problem is $\ln(2\sqrt{2})$.

This entropy loss is in accord with a general result coming from conformal field theory [28, 29]. The entropy of a Hall droplet with trivial total topological charge and perimeter L is $S = \alpha L - \ln \mathcal{D}$, where $\alpha = \pi c T/12$, T is the temperature, c is the central charge of the conformal field theory describing the edge modes, and $\mathcal{D}$ is the total quantum dimension of the particular topological state of matter [30, 31]. When a droplet breaks into two droplets, each of which has trivial topological charge, the entropy of the two droplets is $S = \alpha L_1 + \alpha L_2 - 2\ln \mathcal{D}$, so that the loss in entropy upon breaking is $\ln \mathcal{D}$. The edge theory for the Moore-Read state is the Neveu-Schwarz sector of the second $N = 2$ supersymmetric minimal model [20], which has central charge $c = 3/2$, and quantum dimension $\mathcal{D} = 2\sqrt{2}$. For the $p + ip$ superconductor, the edge theory is the Ising model, which has $c = 1/2$ and $\mathcal{D} = 2$. In both cases, the entropy loss at the point contact is the same as that we deduced above. Physically, the decrease in entropy arises because there is no longer any uncertainty in the topological charge of the two sub-droplets once they break apart. Remarkably, the $T = 0$ entanglement entropy determined from the reduced density matrix of a region

of perimeter L inside a much larger Hall droplet is also of the form, $S = \alpha L - \ln \mathcal{D}$, with α non-universal[30, 31]. This suggests that the entropy loss at the point contact is actually a topological entanglement entropy.

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